

A General Stochastic Maximum Principle For Optimal Control Of Stochastic Systems Driven By Multidimensional Teugel's Martingales[☆]

Jianzhong Lin^{a,*}

^a*Department of Mathematics, Shanghai Jiaotong University, Shanghai 200240, China*

Abstract

A necessary maximum principle is proved for optimal controls of stochastic systems driven by multidimensional Teugel's martingales. The multidimensional Teugel's martingales are constructed by orthogonalizing the multidimensional Lévy processes. The control domain need not be convex, and the control is allowed to enter into the terms of Teugel's martingales.

Keywords: Stochastic optimal control, Maximum principle, Backward stochastic differential equation, Lévy processes, Teugel's martingales.

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1. Introduction

The stochastic maximum principle is one of the central topics in the stochastic optimal control theory. In the past four decades, a variety of results have been obtained on optimal stochastic control problems.(cf. for example, [1], [3], [5], [12], [14]-[17], [26], [31]). Two major advances in these works are worth mentioning. One is the definition of the adjoint processes and its characterization by Itô-type equations. This was contributed by Kushner [17] and Bismut [5], and summarized by Bensoussan [3] via functional analysis methods. Another advance is the idea of second-order variation in calculating the variation of the cost functional caused by the spike variation of the given optimal control. This was motivated by the study of the nonconvex optimal stochastic control of diffusion processes with the control entering into the diffusion term, and was developed by Peng [26]. On nonconvex controls of diffusion processes, we refer the reader to Kushner [17], Haussmann [14], Bensoussan [3], Hu [15], Hu and Peng [16], Peng [26] and Yong and Zhou [35].

It is well known that jump-diffusion process is an important class of processes for describing financial data. The stochastic maximum principle of jump-diffusion processes, where the control is unallowed into the jump terms, was considered by Boel [6], Boel and Varaiya [7], Rishel [28], Davis and Elliott [9] and Situ [31]. The further profound problem, where the control enters into the diffusion and jump terms and also some state constraints are imposed, was completely solved by Tang and Li [34] by applying the idea of second-order variation. On the convex controls of jump-diffusion, we refer the reader to Cadenillas[8], Framstad, Okesendal and Sulem [13], Shi and Wu [30].

The Lévy process (refers to Bertoin [4], Sato [29]) is a more general class of discontinuous processes than jump-diffusion processes. Nualart and Schoutens [22] obtained some interesting results. They introduce the power jump processes and the related Teugel's martingales. Furthermore, they give a chaotic and predictable representation for a one-dimensional Lévy process, in terms of these orthogonalized Teugels martingales. Thus the martingale representation theorem for Lévy process satisfying some exponential moment condition was a consequence of the chaotic representation. Nualart and Schoutens [23] established the existence and uniqueness of solutions for BSDE driven by a one-dimensional Lévy process of the kind considered in Nualart and Schoutens [22]. Further progresses on the

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*Corresponding author.

Email address: jzlin@sjtu.edu.cn (Jianzhong Lin)

subject were subsequently given by Bahlali, Eddahbi and Essaky [2], Ren[27], Lin[20]. Based on these Results, a stochastic linear-quadratic problem with Lévy processes was considered by Mitsui and Tabata [24], Tang and Wu [32]. The stochastic maximum principle, where the control enters into the diffusion and jump terms and also control domain is convex, was given by Meng and Tang [21], Tang and Zhang [33].

Recently, A chaotic and predictable representation theorem associated with multidimensional Lévy processes was obtained by Lin [19]. This extends the setting in Nualart and Schoutens [22] into the multidimensional Lévy processes. Furthermore, The existence and uniqueness of solutions for BSDEs driven by multidimensional Teugel's martingales, which are constructed by orthogonalizing the multidimensional Lévy processes, was proved by Lin [20]. According to these results and following the research line of the paper in Peng [26] and Tang and Li [34], this paper discusses the general stochastic maximum principle where the control systems are driven by the multidimensional Teugel's martingales. It is worth emphasizing that there are three main differences in our setting compared with Mitsui and Tabata [24], Tang and Wu [32], Meng and Tang [21] and Tang and Zhang [33]. First, in our paper, the each component in stochastic system is driven by a Teugel's martingale which is generated by the multidimensional Lévy processes, while the each component in stochastic system in [21], [24], [32] and [33] is driven by a Teugel's martingale which is generated by one component of multidimensional Lévy processes. Secondly, in our paper, the control domain need not be convex, while that in Meng and Tang [21], Tang and Zhang [33] is convex and therefore the second-order variation technique is unnecessary. Finally, the terminal state in our case is constrained while is not in Meng and Tang [21], Tang and Zhang [33].

The paper is organized as follows. Section 2 contains an introduction on chaotic and predictable representation theorem associated with multidimensional Lévy processes and BSDEs driven by multidimensional Teugel's martingales. In Section 3, we give the statement of the problem, our main assumptions and some preliminary lemmas about the first- and second-order variational equation and variational inequality which will be used in the sequel. In Section 4, we derive the first- and second-order adjoint equations, and finally prove the necessary maximum principle. The conclusions are drawn in Section 5.

2. BSDE driven by multidimensional Teugel's martingales

A \mathbb{R}^n -valued stochastic process $X = \{X(t) = (X_1(t), X_2(t), \dots, X_n(t))', t \geq 0\}$ defined in complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called *Lévy process* if X has stationary and independent increments and $X(0) = \mathbf{0}$. A Lévy process possesses a càdlàg modification and we will always assume that we are using this càdlàg version. If we let $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{N}$, where $\mathcal{G}_t = \sigma\{X(s), 0 \leq s \leq t\}$ is the natural filtration of X , and \mathcal{N} are the \mathbb{P} -null sets of \mathcal{F} , then $\{\mathcal{F}_t, t \geq 0\}$ is a right continuous family of σ -fields. We assume that \mathcal{F} is generated by X . For an up-to-date and comprehensive account of Lévy processes we refer the reader to Bertoin [4] and Sato [29].

Let X be a Lévy process and denote by

$$X(t-) = \lim_{s \rightarrow t-, s < t} X(s), \quad t > 0,$$

the left limit process and by $\Delta X(t) = X(t) - X(t-)$ the jump size at time t . It is known that the law of $X(t)$ is *infinitely divisible* with characteristic function of the form

$$E[\exp(i\theta \cdot X(t))] = (\phi(\theta))^t, \quad \theta = (\theta_1, \theta_2, \dots, \theta_n) \in \mathbb{R}^n$$

where $\phi(\theta)$ is the characteristic function of $X(1)$. The function $\psi(\theta) = \log \phi(\theta)$ is called the *characteristic exponent* and it satisfies the following famous Lévy-Khintchine formula (Bertoin, [4]):

$$\psi(\theta) = -\frac{1}{2}\theta \cdot \Sigma \theta + i\mathbf{a} \cdot \theta + \int_{\mathbb{R}^n} (\exp(i\theta \cdot \mathbf{x}) - 1 - i\theta \cdot \mathbf{x}1_{|\mathbf{x}| \leq 1}) \nu(d\mathbf{x}).$$

where $\mathbf{a}, \mathbf{x} \in \mathbb{R}^n$, Σ is a symmetric nonnegative-definite $n \times n$ matrix, and ν is a measure on $\mathbb{R}^n \setminus \{o\}$ with $\int (||\mathbf{x}||^2 \wedge 1) \nu(d\mathbf{x}) < \infty$. The measure ν is called the *Lévy measure* of X .

Throughout this paper, we will use the standard multi-index notation. We denote by \mathbb{N}_0 the set of nonnegative integers. A multi-index is usually denoted by \mathbf{p} , $\mathbf{p} = (p_1, p_2, \dots, p_n) \in \mathbb{N}_0^n$. Whenever \mathbf{p} appears with subscript or superscript, it means a multi-index. In this spirit, for example, for $\mathbf{x} = (x_1, \dots, x_n)$, a monomial in variables x_1, \dots, x_n

is denoted by $\mathbf{x}^{\mathbf{p}} = x_1^{p_1} \cdots x_n^{p_n}$. In addition, we also define $\mathbf{p}! = p_1! \cdots p_n!$ and $|\mathbf{p}| = p_1 + \cdots + p_n$; and if $\mathbf{p}, \mathbf{q} \in \mathbb{N}_0^n$, then we define $\delta_{\mathbf{p}, \mathbf{q}} = \delta_{p_1, q_1} \cdots \delta_{p_n, q_n}$.

In the remaining of the paper, we will suppose that

Assumption 2.1. *the Lévy measure satisfies for some $\varepsilon > 0$, and $\lambda > 0$,*

$$\int_{|\mathbf{x}| \geq \varepsilon} \exp(\lambda \|\mathbf{x}\|) \nu(d\mathbf{x}) < \infty.$$

This implies that

$$\int \mathbf{x}^{\mathbf{p}} \nu(d\mathbf{x}) < \infty, \quad |\mathbf{p}| \geq 2$$

and that the characteristic function $E[\exp(i\boldsymbol{\theta} \cdot X(t))]$ is analytic in a neighborhood of origin \mathbf{o} . As a consequence, $X(t)$ has moments of all orders and the polynomials are dense in $L^2(\mathbb{R}^n, \mathbb{P} \circ X(t)^{-1})$ for all $t > 0$.

Fix a time interval $[0, T]$ and set $L_T^2 = L^2(\Omega, \mathcal{F}_T, \mathbb{P})$. We will denote by \mathcal{P} the predictable sub- σ -field of $\mathcal{F}_T \otimes \mathcal{B}_{[0, T]}$. First we introduce some notation:

- : Let H_T^2 denote the space of square integrable and \mathcal{F}_t -progressively one-dimensional measurable processes $\phi = \{\phi(t), t \in [0, T]\}$ such that

$$\|\phi\|^2 = \mathbb{E} \left[\int_0^T \|\phi(t)\|^2 dt \right] < \infty.$$

- : M_T^2 will denote the subspace of H_T^2 formed by predictable processes.
- : $(H_T^2(l^2))^m$ and $(M_T^2(l^2))^m$ are the corresponding spaces of m -dimensional l^2 -valued processes equipped with the norm

$$\begin{aligned} \|\phi_k\|_{l^2}^2 &= \mathbb{E} \left[\int_0^T \sum_{d=1}^{\infty} \sum_{\mathbf{p} \in \mathbb{N}_d^n} |\phi_k^{\mathbf{p}}|^2 \right] \quad k = 1, 2, \dots, m, \\ \|\phi\|_{(l^2)^m}^2 &= \sum_{k=1}^m \|\phi_k(t)\|_{l^2}^2, \end{aligned}$$

where $\phi = (\phi_1, \phi_2, \dots, \phi_m)'$, $\phi_k = \{\phi_k^{\mathbf{p}} : \mathbf{p} \in \mathbb{N}_d^n\}$, $k = 1, 2, \dots, m$ and $\mathbb{N}_d^n \stackrel{\text{def}}{=} \{\mathbf{p} \in \mathbb{N}_0^n : |\mathbf{p}| = d\}$.

- : Set $\mathcal{H}_T^2 = H_T^2 \times (M_T^2(l^2))^m$.

Following Lin [19] we introduce power jump monomial processes of the form

$$X(t)^{(p_1, \dots, p_n)} \stackrel{\text{def}}{=} \sum_{0 \leq s \leq t} (\Delta X_1(s))^{p_1} \cdots (\Delta X_n(s))^{p_n},$$

The number $|\mathbf{p}|$ is called the total degree of $X(t)^{\mathbf{p}}$. Furthermore define

$$Y(t)^{(p_1, \dots, p_n)} \stackrel{\text{def}}{=} X(t)^{(p_1, \dots, p_n)} - \mathbb{E}[X(t)^{(p_1, \dots, p_n)}] = X(t)^{(p_1, \dots, p_n)} - m_{\mathbf{p}} t,$$

the compensated power jump process of multi-index $\mathbf{p} = (p_1, p_2, \dots, p_n)$ where $m_{\mathbf{p}} = \int \prod_{i=1}^n x_i^{p_i} \nu(d\mathbf{x})$. Under hypothesis 1, $Y(t)^{(p_1, \dots, p_n)}$ is a normal martingale, since for an integrable Lévy process Z , the process $\{Z_t - E[Z_t], t \geq 0\}$ is a martingale. We call $Y(t)^{(p_1, \dots, p_n)}$ the *Teugels martingale monomial* of multi-index (p_1, \dots, p_n) .

We can apply the standard Gram-Schmidt process with the graded lexicographical order to generate a biorthogonal basis $\{H^p, p \in \mathbb{N}^n\}$, such that each $H^p(|p| = d)$ is a linear combination of the Y^q , with $|q| \leq |p|$ and the leading coefficient equal to 1. We set

$$H^p = Y^p + \sum_{q < p, |q|=|p|} c_q Y^q + \sum_{k=1}^{|p|-1} \sum_{|q|=k} c_q Y^q,$$

where $p = \{p_1, \dots, p_n\}$, $q = \{q_1, \dots, q_n\}$ and $<$ represent the relation of graded lexicographical order between two multi-indexes. Some details about the technique and theory of orthogonal polynomials of several variables refer to Dunkl and Xu [11].

Set

$$\begin{aligned} p(x)^p &= x^p + \sum_{q < p, |q|=|p|} c_q x^q + \sum_{k=1}^{|p|-1} \sum_{|q|=k} c_q x^q, \\ \tilde{p}(x)^p &= x^p + \sum_{q < p, |q|=|p|} c_q x^q + \sum_{k=2}^{|p|-1} \sum_{|q|=k} c_q x^q, \end{aligned}$$

Set

$$\begin{aligned} H^p(t) &= \sum_{0 \leq s \leq t} \left((\Delta X_1)^{p_1} \dots (\Delta X_n)^{p_n} + \sum_{q < p, |q|=|p|} c_q (\Delta X_1)^{q_1} \dots (\Delta X_n)^{q_n} \right. \\ &\quad \left. + \sum_{k=1}^{|p|-1} \sum_{|q|=k} c_q (\Delta X_1)^{q_1} \dots (\Delta X_n)^{q_n} \right) \\ &\quad - t \mathbb{E} \left[X^p(1) + \sum_{q < p, |q|=|p|} c_q X^q(1) + \sum_{k=1}^{|p|-1} \sum_{|q|=k} c_q X^q(1) \right] \\ &= (c_{e_1} X_1(1) + \dots + c_{e_n} X_n(1)) + \sum_{0 \leq s \leq t} \tilde{p}(\Delta X(s)) \\ &\quad - t \mathbb{E} \left[\sum_{0 \leq s \leq t} \tilde{p}(\Delta X(s)) \right] - t \mathbb{E} [c_{e_1} X_1(1) + \dots + c_{e_n} X_n(1)]. \end{aligned}$$

Specially we have

$$\begin{aligned} H^{e_1}(t) &= c_{e_1}(1)(X_1(t) - t\mathbb{E}(X_1(1))), \\ H^{e_2}(t) &= c_{e_2}(2)(X_2(t) - t\mathbb{E}(X_2(1))) + c_{e_1}(2)(X_1(t) - t\mathbb{E}(X_1(1))), \\ &\vdots \\ H^{e_n}(t) &= c_{e_n}(n)(X_n(t) - t\mathbb{E}(X_n(1))) + c_{e_{n-1}}(n)(X_{n-1}(t) - t\mathbb{E}(X_{n-1}(1))) \\ &\quad + \dots + c_{e_1}(n)(X_1(t) - t\mathbb{E}(X_1(1))). \end{aligned} \tag{2.1}$$

The main tool in the theory of BSDEs is the martingale representation theorem (cf. Pardoux and Peng [25]). Nualart and Schoutens [22] had proved the representation theorem associated with one-dimensional Lévy process, furthermore Nualart and Schoutens [23] had established the existence and uniqueness of solutions for BSDE driven by a one-dimensional Teugel's martingale generated by the Lévy process. The main results in Lin [19] is the Predictable Representation Property (PRP) associated multidimensional Lévy processes:

Lemma 2.1. *Every random variable F in $L^2(\Omega, \mathcal{F})$ has a representation of the form*

$$F = \mathbb{E}(F) + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_d^n} \int_0^T \Phi^p(s) dH^p(s)$$

where $\Phi^p(s)$ is predictable. This result is an extended version for the corresponding Theorem in Nualart and Schouten [22].

Taking into account the results and notation presented in the previous section, it seems natural to consider the BSDEs with the following form

$$-dY(t) = f(t, Y(t-), Z(t))dt - \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_d^n} z^p(s) dH^p(s), \quad Y(T) = \xi, \quad (2.2)$$

where

- $Y(t) = (Y_1(t), Y_2(t), \dots, Y_m(t))'$.
- $Z(t) = \{z^p(t)\}_{p \in \mathbb{N}_0^n}$, each component $z^p(t) = (z_1^p, \dots, z_m^p)'$ is a m -variables \mathcal{F}_t predictable function;
- $f = (f_1, f_2, \dots, f_m)' : \Omega \times [0, T] \times \mathbb{R}^m \times (M_T^2(l^2))^m \rightarrow \mathbb{R}^m$ is a measurable m -dimensional vector function such that $f(\cdot, \mathbf{0}, \mathbf{0}) \in (H_T^2)^m$.
- f is uniformly Lipschitz in the first two components, i.e., there exists $C_k > 0, k = 1, 2, \dots, m$, such that $dt \otimes d\mathbb{P}$ a.s., for all (y_1, z_1) and (y_2, z_2) in $\mathbb{R}^m \times (l^2)^m$

$$|f_k(t, y_1, z_1) - f_k(t, y_2, z_2)| \leq C_k (\|y_1 - y_2\|_2 + \|z_1 - z_2\|_{(l^2)^m}), \quad k = 1, 2, \dots, m.$$

- $\xi \in L_T^2(\Omega, \mathbb{P})$.

If (f, ξ) satisfies the above assumptions, the pair (f, ξ) is said to be **standard data** for BSDE. A solution of the BSDE is a pair of processes, $\{(Y(t), Z(t)), 0 \leq t \leq T\} \in H_T^2 \times (M_T^2(l^2))^m$ such that the following relation holds for all $t \in [0, T]$:

$$Y(t) = \xi + \int_t^T f(s, Y(s-), Z(s))ds - \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_d^n} \int_t^T z^p(s) dH^p(s). \quad (2.3)$$

A key-result concerning the existence uniqueness of solution of BSDEs (2.2) is given by Lin [20]:

Lemma 2.2. *Given standard data (f, ξ) , there exists a unique solution (Y, Z) which solves the BSDE (2.3)*

3. Notations and preliminary lemmas

Consider the following stochastic control system:

$$\begin{aligned} dx(t) &= g(x(t-), v(t))dt + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_d^n} \gamma^p(x(t-), v(t))dH^p(t), \\ x(0) &= x_0. \end{aligned} \quad (3.1)$$

Here and hereafter

$$\begin{aligned} g(x, v) &: \mathbb{R}^m \times \mathcal{U} \rightarrow \mathbb{R}^m, \\ \gamma^p(x, v) &: \mathbb{R}^m \times \mathcal{U} \rightarrow \mathbb{R}^m, \forall p \in \mathbb{N}^n, \end{aligned}$$

and \mathcal{U} is a nonempty subset of \mathbb{R}^m (control domain). An admissible control $v(\cdot)$ is a \mathcal{F}_t -predictable process with values in \mathcal{U} such that

$$\|v(\cdot)\| =: \sup_{0 \leq t \leq T} \left[E|v(t)|^8 \right]^{\frac{1}{8}} < \infty \quad (3.2)$$

We denote the set of all admissible controls by \mathcal{U}_{ad} . When $\mathcal{U} = \mathbb{R}^m$, we write $L_{\mathcal{F},p}^{\infty,8}[[0, 1]; \mathbb{R}^m]$ for \mathcal{U}_{ad} . The terminal constraint is

$$\mathbb{E}G(x_0, X(T)) \in Q \subset \mathbb{R}^k, \quad (3.3)$$

where $G(\cdot, \cdot) =: (G^1(\cdot, \cdot), \dots, G^k(\cdot, \cdot))$ and $G^i(\cdot, \cdot) : \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}^k$ for $i = 1, 2, \dots, k$.

The cost functional is

$$J(v(\cdot), x_0) = E \int_0^T \ell(x(t), v(t)) dt + Eh(x_0, x(T)), \quad (3.4)$$

where

$$\ell(x, v) : \mathbb{R}^m \times \mathcal{U} \rightarrow \mathbb{R}, \quad h(x) : \mathbb{R}^m \rightarrow \mathbb{R}.$$

Our optimal control problem is to find a pair $(y_0, u(\cdot)) \in \mathbb{R}^m \times \mathcal{U}_{ad}$ such that (3.1) and (3.3) are satisfied and (3.4) is minimized

Throughout the paper, we make the following assumptions

Assumption 3.1. *The vector functions $g(x, v)$, $G(y, x)$, $\ell(x, v)$, $h(y, x)$ and $\gamma^p(x, v)$ ($p \in \mathbb{N}_0^n$) are twice continuously differentiable with respect to x , and $G(y, x)$, $h(y, x)$ are differentiable in y . They and their derivatives in x or y are continuous in (x, v) and (y, x) . The vector functions $g(x, v)$, $G_{y_i}(y, x)$, $G_{x_i}(y, x)$, $\ell_{x_i}(x, v)$, $h_{y_i}(y, x)$, $h_{x_i}(y, x)$, and*

$$\left[\sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_d^n} |\mathbf{z}^p(x, v)|^{2k} \right]^{\frac{1}{2k}}, \quad k = 1, 2,$$

($i = 1, \dots, n$), are bounded by $(1 + |x| + |y| + |v|)$. The vector functions $G(y, x)$, $\ell(x, v)$, $h(y, x)$ are bounded by $(1 + |x|^2 + |y|^2 + |v|^2)$, $g_{x_i}(x, v)$, $g_{x_i x_j}(x, v)$, $G_{x_i x_j}(y, x)$, $\ell_{x_i x_j}(x, v)$, $h_{x_i x_j}(y, v)$, and

$$\sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_d^n} |\mathbf{z}_{x_i}^p(x, v)|^{2k}, \quad k = 1, 2, \quad \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_d^n} |\mathbf{z}_{x_i x_j}^p(x, v)|^2$$

($i, j = 1, \dots, n$) are bounded. Here x_i, y_i ($i = 1, \dots, n$) stand for the i th coordinates of x and y respectively.

Assumption 3.2. *The set Q is closed and convex.*

Let $(y_0, y(\cdot), u(\cdot))$ be an optimal triplet of the problem. For the given $(x_0, v(\cdot)) \in \mathbb{R}^m \times \mathcal{U}_{ad}$, write $y(\cdot; v(\cdot), x_0)$ for the solution of (3.1). For $v(\cdot)$, $v_1(\cdot)$, $v_2(\cdot) \in \mathcal{U}_{ad}$, denote

$$\begin{aligned} \Delta m(s; v_2, v_1) &\stackrel{\text{def}}{=} m(y(s-), v_2) - m(y(s-), v_1), \\ \Delta m(s; v) &\stackrel{\text{def}}{=} m(y(s-), v) - m(y(s-), u(s)), \\ m(s; v_1) &\stackrel{\text{def}}{=} m(y(s), v_1), \\ m(s) &\stackrel{\text{def}}{=} m(y(s), u(s)), \end{aligned} \quad (3.5)$$

with m standing for g, γ, ℓ and all their (up to second-) derivatives in x .

For $I_0 \subset [0, 1]$, let $|I_0|$ denote the Lebesgue measure of the set I_0 . Let $v(\cdot)$, $v_1(\cdot)$, $v_2(\cdot) \in \mathcal{U}_{ad}$. Define

$$\hat{d}(v_1(\cdot), v_2(\cdot)) \stackrel{\text{def}}{=} |\{t \in [0, 1]; E|v_1(\cdot) - v_2(\cdot)|^2 > 0\}|. \quad (3.6)$$

For $\rho \in (0, T]$, $I_\rho \subset [0, T]$ and $v(\cdot) \in \mathcal{U}_{ad}$, It is classical to construct a perturbed admissible control in the following way (spike variation):

$$\begin{aligned} u^\rho(s) &\stackrel{\text{def}}{=} u(s)\chi_{[0,1] \setminus I_\rho}(s) + v(s)\chi_{I_\rho}(s), \quad s \in [0, T], \\ y_0^\rho &\stackrel{\text{def}}{=} y_0 + |I_\rho|\eta, \quad \eta \in \mathbb{R}^n \\ y^\rho(\cdot) &\stackrel{\text{def}}{=} y(\cdot; u^\rho(\cdot), y_0^\rho), \end{aligned} \quad (3.7)$$

with $\chi_A(\cdot)$ denoting the indicator function of some set A . Obviously, we have

$$\hat{d}(u_\rho(\cdot), u(\cdot)) = |I_\rho|. \quad (3.8)$$

We can prove that $u^\rho(\cdot) \in \mathcal{U}_{ad}$.

Lemma 3.1. *Let the Assumption 3.1 hold. Then for $v(\cdot), u(\cdot), u^\rho(\cdot) \in \mathcal{U}_{ad}$*

$$\begin{aligned} \sup_{t \in [0, T]} \mathbb{E} |y(t; v(\cdot), x_0)|^8 &= O((1 + \|v(\cdot)\|)^8), \\ \sup_{t \in [0, T]} \mathbb{E} |y(t, u(\cdot), y_0) - y(t; u^\rho(\cdot), x_0)|^4 &= O(\hat{d}^2(u^\rho(\cdot), u(\cdot))(1 + \|u(\cdot)\| + \|u^\rho(\cdot)\|)^4), \\ \sup_{t \in [0, T]} \mathbb{E} |y_1(t; u^\rho(\cdot), u(\cdot))|^8 &= O(\hat{d}^4(u^\rho(\cdot), u(\cdot))(1 + \|u(\cdot)\| + \|u^\rho(\cdot)\|)^8), \\ \sup_{t \in [0, T]} \mathbb{E} |y_2(t; u^\rho(\cdot), u(\cdot))|^4 &= O(\hat{d}^4(u^\rho(\cdot), u(\cdot))(1 + \|u(\cdot)\| + \|u^\rho(\cdot)\|)^8), \\ \sup_{t \in [0, T]} \mathbb{E} |y(t; u^\rho, y_0 + \hat{d}(u_i, u)\eta) - y(t; u, y_0) - y_1(t; u^\rho, u) - y_2(t; u^\rho, u)|^2 \\ &= O(\hat{d}^2(u^\rho(\cdot), u(\cdot))(1 + \|u(\cdot)\| + \|u^\rho(\cdot)\|)^8), \quad \text{as } \hat{d}(u^\rho(\cdot), u(\cdot)) \rightarrow 0. \end{aligned} \quad (3.9)$$

where $y_1(\cdot), y_2(\cdot)$ are the solutions of

$$\begin{aligned} y_1(t) &= \int_0^t g_x(y(s), u(s))y_1(s)ds \\ &\quad + \sum_{d=1}^\infty \sum_{p \in \mathbb{N}_d^n} \int_0^t \left[\gamma_x^p(y(s), u(s))y_1(s) + \Delta \gamma^p(s, u^\rho(s), u(s)) \right] dH^p(s) \end{aligned} \quad (3.10)$$

$$\begin{aligned} y_2(t) &= \hat{d}(u^\rho(\cdot), u(\cdot))\eta \\ &\quad + \int_0^t \left[g_x(y(s), u(s))y_2(s) + \Delta g(s, u^\rho(s), u(s)) + \frac{1}{2}g_{xx}(y(s), u(s))y_1(s)y_1(s) \right] ds \\ &\quad + \sum_{d=1}^\infty \sum_{p \in \mathbb{N}_d^n} \int_0^t \left[\gamma_x^p(y(s), u(s))y_2(s) + \frac{1}{2}\gamma_{xx}^p(y(s), u(s))y_1(s)y_1(s) \right] dH^p(s) \\ &\quad + \sum_{d=1}^\infty \sum_{p \in \mathbb{N}_d^n} \int_0^t \Delta \gamma_x^p(s, u^\rho(s), u(s))y_1(s) dH^p(s) \end{aligned} \quad (3.11)$$

where $f_{xx}yy = \sum_{i,j=1}^m f_{x^i x^j} y^i y^j$ for $f = g, \gamma^p$.

Proof. Without loss of generality, we assume $\eta = 0$. Define

$$\int_{I_\rho} g_0(s) dH^p(s) =: \int \chi_{I_\rho}(s) g_0(s) dH^p(s). \quad \forall p \in \mathbb{N}^n$$

We have the following inequalities for $p > 1$:

$$\begin{aligned} \mathbb{E} \left| \int_{I_\rho} f_0(s) ds \right|^p &\leq C_p |I_\rho|^{p-1} \mathbb{E} \int_{I_\rho} |f_0(s)|^p ds, \\ \mathbb{E} \left| \int_{I_\rho} g_0(s) dH^p(s) \right|^{2p} &\leq C_p |I_\rho|^{p-1} \mathbb{E} \int_{I_\rho} |g_0(s, z)|^{2p} ds, \quad \forall p \in \mathbb{N}^n. \end{aligned} \quad (3.12)$$

By virtue of the Assumption 3.1, we have

$$\begin{aligned} \sup_{t \in [0, T]} \mathbb{E} |y(t)|^8 &= O((1 + \|v(\cdot)\| + \|u(\cdot)\|)^8), \\ \sup_{t \in [0, T]} \mathbb{E} |\Delta g(t, u^\rho(s))|^4 &= O((1 + \|u^\rho(\cdot)\| + \|u(\cdot)\|)^4), \\ \sup_{t \in [0, T]} \mathbb{E} |\Delta \gamma^p(t; u^\rho(s))|^8 &= O((1 + \|u^\rho(\cdot)\| + \|u(\cdot)\|)^8), \quad \forall p \in \mathbb{N}^n. \end{aligned} \quad (3.13)$$

Then we can obtain the following inequalities by using (3.12):

$$\begin{aligned} \mathbb{E} \left| \int_0^T \Delta g(t, u^\rho(s)) \right|^4 &= O(|I_\rho|^4 (1 + \|v(\cdot)\| + \|u(\cdot)\|)^4), \\ \mathbb{E} \left| \int_0^T \Delta \gamma^p(t; u^\rho(\cdot)) \right|^8 &= O(|I_\rho|^4 (1 + \|v(\cdot)\| + \|u(\cdot)\|)^8), \quad \forall p \in \mathbb{N}^n. \end{aligned} \quad (3.14)$$

Then the first four estimates of (3.9) are easily proved by using the familiar elementary inequalities

$$(m_1 + m_2)^i \leq C(|m_1|^i + |m_2|^i), i = 4, 8$$

and the well-known Gronwall's inequality.

The proof for the last estimate follows. Set $y_3 = y_1 + y_2$. We have

$$\begin{aligned} & \int_0^t g(y + y_3, u^\rho) ds + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_d^n} \int_0^t \gamma^p(y + y_3, u^\rho) dH^p(s), \\ &= \int_0^t \left[g(y, u^\rho) + g_x(y, u^\rho) y_3 + \int_0^1 \int_0^1 \lambda g_{xx}(y + \lambda \mu y_3, u^\rho) d\lambda d\mu y_3 y_3 \right] ds \\ & \quad + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_d^n} \int_0^t \left[\gamma^p(y, u^\rho) + \gamma_x^p(y, u^\rho) y_3 + \int_0^1 \int_0^1 \lambda \gamma_{xx}^p(y + \lambda \mu y_3, u^\rho) d\lambda d\mu y_3 y_3 \right] dH^p(s) \\ &= \int_0^t g(y, u) ds + \int_0^t g_x(y, u) y_3 ds + \int_0^t \Delta g(s, u^\rho(s), u(s)) ds \\ & \quad + \int_0^t \Delta g_x(s, u^\rho(s), u(s)) y_3(s) ds + \int_0^t \frac{1}{2} g_{xx}(y, u) y_3(s) y_3(s) ds \\ & \quad + \int_0^t \int_0^1 \int_0^1 \lambda [g_{xx}(y + \lambda \mu y_3, u^\rho) - g_{xx}(y, u)] d\lambda d\mu y_3 y_3 ds \\ & \quad + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_d^n} \int_0^t \gamma^p(y, u) dH^p(s) + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_d^n} \int_0^t \gamma_x^p(y, u) y_3 dH^p(s) \\ & \quad + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_d^n} \int_0^t \gamma^p(s, u^\rho(s), u(s)) dH^p(s) + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_d^n} \int_0^t \Delta \gamma_x^p(s, u^\rho(s), u(s)) y_3 dH^p(s) \\ & \quad + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_d^n} \int_0^t \frac{1}{2} \gamma_{xx}^p(y, u) y_3(s) y_3(s) dH^p(s) \\ & \quad + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_d^n} \int_0^t \int_0^1 \int_0^1 \lambda [\gamma_{xx}^p(y + \lambda \mu y_3, u^\rho) - \gamma_{xx}^p(y, u)] y_3 y_3 d\lambda d\mu dH^p(s) \\ &= y(t) + y_3(t) - y_0 + \int_0^t G^\rho(s) ds + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_d^n} \int_0^t \Xi^{\rho, p}(s) dH^p(s), \end{aligned}$$

where

$$\begin{aligned}
G^\rho(s) &= \frac{1}{2}g_{xx}(y(s), u(s))(y_2(s)y_2(s) + 2y_1(s)y_2(s)) \\
&\quad + \Delta g_x(y(s), u^\rho(s), u(s))y_2(s) \\
&\quad + \int_0^1 \int_0^1 \lambda [g_{xx}(y + \lambda \mu y_3, u^\rho) - g_{xx}(y, u)] d\lambda d\mu y_3(s)y_3(s) \\
\Xi^{\rho, P}(s) &= \frac{1}{2}\gamma_{xx}^P(y(s), u(s))(y_2(s)y_2(s) + 2y_1(s)y_2(s)) \\
&\quad + \Delta \gamma_x^P(y(s), u^\rho(s), u(s))y_2(s) \\
&\quad + \int_0^1 \int_0^1 \lambda [\gamma_{xx}^P(y + \lambda \mu y_3, u^\rho) - \gamma_{xx}^P(y, u)] d\lambda d\mu y_3(s)y_3(s)
\end{aligned}$$

Since

$$\begin{aligned}
y(t) + y_3(t) &= y_0 + \int_0^t g(y + y_3, u^\rho)ds + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_d^n} \int_0^t \gamma^p(y + y_3, u^\rho) dH^p(s) \\
&\quad - \int_0^t G^\rho(s)ds - \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_d^n} \int_0^t \Xi^{\rho, P}(s) dH^p(s).
\end{aligned}$$

and

$$y^\rho(t) = y_0 + \int_0^t g(y^\rho(s), u^\rho(s))ds + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_d^n} \int_0^t \gamma^p(y^\rho(s), u^\rho(s)) dH^p(s),$$

we can derive that

$$\begin{aligned}
(y^\rho - y - y_3)(t) &= \int_0^t A^\rho(s)(y^\rho - y - y_3)(s)ds \\
&\quad + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_d^n} \int_0^t F^{\rho, P}(s)(y^\rho - y - y_3)(s) dH^p(s) \\
&\quad + \int_0^t G^\rho(s)ds + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_d^n} \int_0^t \Xi^{\rho, P}(s) dH^p(s).
\end{aligned}$$

$$|A^\rho(s, \omega)| + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_d^n} |F^{\rho, P}(s, \omega)| \leq C \quad \forall s, \quad \forall \omega.$$

and

$$\sup_{0 \leq t \leq T} E \left(\left| \int_0^t G^\rho(s)ds \right|^2 + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_d^n} \left| \int_0^t \Xi^{\rho, P}(s) dH^p(s) \right|^2 \right) = o(|I_\rho|^2(1 + \|u^\rho(\cdot)\| + \|u(\cdot)\|)^8).$$

From these we can use Itô's formula and Gronwall's inequality to obtain the fifth estimate (3.9). The proof is completed. \square

Lemma 3.2. Assume that $l(\cdot)$ is a scalar-valued Lebesgue integrable function defined on $[0, T]$. Then for $\rho \in (0, T]$, there exists a measurable subset $I_\rho \subset [0, T]$, such that

$$\begin{aligned}
|I_\rho| &= \rho, \\
\int_{I_\rho} l(s)ds &= \rho \int_{[0, T]} l(s)ds + o(\rho), \quad \rho \rightarrow 0.
\end{aligned} \tag{3.15}$$

The proof is quite elementary and the reader is referred to [18].

4. Adjoint equations and the maximum principle

The Hamiltonian is defined as

$$H(x, v, \lambda, p, J) = \lambda \ell(x, v) + (p, g(x, v)) + \sum_{i=1}^{\infty} \sum_{p \in \mathbb{N}_i^r} (J^p, \gamma^p(x, v)).$$

this is a map from $\mathbb{R}^m \times \mathcal{U} \times \mathbb{R} \times \mathbb{R}^m \times (M_T^2(l^2))^m$ into \mathbb{R} . Here we have used (\cdot, \cdot) for the scalar product of Euclidean spaces.

From Lemma 2.2 and Assumption 3.1, we see for the given $p(T) \in L^2(\Omega, \mathcal{F}_T; \mathbb{R}^m)$, $P(T) \in L^2(\Omega, \mathcal{F}_T; \mathbb{R}^{m \times m})$ that the Itô-type adjoint equations

$$\begin{aligned} -dp(t) &= \left[g_x^\top(y(t), u(t))p(t) + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_d^n} \gamma_x^p(y(t), u(t))^\top J^p(t) + \lambda \ell_x(y(t), u(t)) \right] dt \\ &\quad - \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_d^n} J^p(t) dH^p(t) \\ p(T) &= h_x(y(T)). \end{aligned} \quad (4.1)$$

and

$$\begin{aligned} -dP(t) &= \left[g_x^\top(y(t), u(t))P(t) + P(t)g_x(y(t), u(t)) + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_d^n} \gamma_x^p(y(t), u(t))^\top P(t) \gamma_x^p(y(t), u(t)) \right. \\ &\quad + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_d^n} \gamma_x^p(y(t), u(t)) R^p(t) \\ &\quad \left. + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_d^n} R^p(t) \gamma_x^p(y(t), u(t)) + H_{xx}(y(t), u(t), \lambda, p(t), J(t)) \right] dt \\ &\quad - \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_d^n} R^p(t) dH^p(t) \\ P(T) &= h_{xx}(y(T)) \end{aligned} \quad (4.2)$$

admit unique solutions $(p(\cdot), \{J^p(\cdot)\}_{p \in \mathbb{N}^n})$ and $(P(\cdot), \{R^p(\cdot)\}_{p \in \mathbb{N}^n})$, with $p(\cdot)$ and $P(\cdot)$ being cadlag processes.

Define the following function:

$$\Phi(s, z; \varepsilon) \stackrel{\text{def}}{=} \inf_{(t, \bar{z}) \in (-\infty, J(u(\cdot), y_0) - \varepsilon] \times Q} \sqrt{|t - s|^2 + |\bar{z} - z|^2} \quad (4.3)$$

Tang and Li [34] had proved the following result.

Lemma 4.1. *For given $\varepsilon > 0$, the function $\Phi(s, z; \varepsilon)$ is continuously differentiable on the open set $\hat{Q} \stackrel{\text{def}}{=} \{(s, z) : \Phi(s, z; \varepsilon) > 0\}$. Moreover, when $\Phi(s, z; \varepsilon) > 0$, we have*

$$\begin{aligned} \langle \Phi_z(s, z; \varepsilon), \hat{z} - z \rangle &\leq 0, \quad \forall \hat{z} \in Q, \\ \Phi_s(s, z; \varepsilon) &\geq 0, \\ |\Phi_s(s, z; \varepsilon)|^2 + |\Phi_z(s, z; \varepsilon)|^2 &= 1. \end{aligned} \quad (4.4)$$

They introduce the smooth function $\alpha(\cdot)$ defined by

$$\alpha(t, z) \stackrel{\text{def}}{=} \begin{cases} C \exp(t^2 + |z|^2 - 1)^{-1}, & t^2 + |z|^2 < 1, \\ 0, & t^2 + |z|^2 \geq 1. \end{cases}$$

Choose the constant C such that

$$\int_{\mathbb{R} \times \mathbb{R}^k} \alpha(t, z) dt dz = 1.$$

Set

$$\alpha_\delta(t, z) = \delta^{-(k+1)} \alpha\left(\frac{t}{\delta}, \frac{z}{\delta}\right). \quad (4.5)$$

They also define the smooth approximation $\Psi(\cdot, \cdot; \varepsilon, \delta)$ of $\Phi(\cdot, \cdot; \varepsilon)$ as follows:

$$\Psi(s, z; \varepsilon, \delta) \stackrel{\text{def}}{=} \int_{\mathbb{R} \times \mathbb{R}^k} \Phi(s - \bar{s}, z - \bar{z}; \varepsilon) \alpha_\delta(\bar{s}, \bar{z}) d\bar{s} d\bar{z} = 1. \quad (4.6)$$

Then it is easy to have

$$0 \leq \Psi(J(u(\cdot), y_0), EG(y_0, y(T)); \varepsilon, \delta) \leq \varepsilon + \sqrt{2}\delta$$

Moreover, Tang and Li [34] gave the following lemma.

Lemma 4.2. *For \hat{Q} defined in Lemma 4.1, we have for $(s, z) \in \hat{Q}$,*

$$\begin{aligned} \lim_{\delta \rightarrow 0+} \Psi_s(s, z; \varepsilon, \delta) &= \Phi_s(s, z; \varepsilon), \\ \lim_{\delta \rightarrow 0+} \Psi_z(s, z; \varepsilon, \delta) &= \Phi_z(s, z; \varepsilon). \end{aligned} \quad (4.7)$$

Our main result in this paper is almost similar to that in Tang and Li [34] in many places:

Theorem 4.1. *Assume Assumptions 3.1 and 3.2 hold. Let $(y_0, y(\cdot), u(\cdot))$ be an optimal triplet. Then there exist*

$$\begin{aligned} 0 \leq \lambda \in \mathbb{R}, \quad \mu &\stackrel{\text{def}}{=} \{\mu^j\}_1^k \in \mathbb{R}^k, \\ (p(\cdot), J(\cdot)) &\in L_{\mathcal{F}}^2(0, T; \mathbb{R}^m) \times L_{\mathcal{F}}^2(0, T; (M_T^2(l^2))^m) \\ (P(\cdot), R(\cdot)) &\in L_{\mathcal{F}}^2(0, T; \mathbb{R}^{m \times m}) \times L_{\mathcal{F}}^2(0, T; (M_T^2(l^2))^{m \times m}) \end{aligned}$$

such that we have the following.

1) *The nontrivial condition*

$$|\lambda|^2 + |\mu|^2 = 1, \quad (4.8)$$

is satisfied.

2) *The Itô-type adjoint equations (4.1), (4.2), as well as*

$$\begin{cases} p(T) &= \lambda h_x(y_0, y(T)) + \sum_{j=1}^k \mu^j G_x^j(y_0, y(T)), \\ p(0) &= -\lambda E h_x(y_0, y(T)) - \sum_{j=1}^k \mu^j E G_y^j(y_0, y(T)) \end{cases} \quad (4.9)$$

and

$$p(T) = \lambda h_{xx}(y_0, y(T)) + \sum_{j=1}^k \mu^j G_{xx}^j(y_0, y(T)), \quad (4.10)$$

are satisfied, with $p(\cdot)$ and $P(\cdot)$ being cadlag processes.

3) The following maximum condition holds:

$$\begin{aligned}
& H(y(s-), v, \lambda, p(s-), J(s)) - H(y(s-), u(s), \lambda, K(s), J(s)) \\
& + \frac{1}{2} \text{tr} P(s-) \left[\sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_d^n} \Delta \gamma^p(s; v) \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_d^n} \Delta \gamma^{p^\top}(s; v) \right] \\
& + \frac{1}{2} \text{tr} \left[\sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_d^n} R^p(s) \right]^\top \left[\sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_d^n} \Delta \gamma^p(s; v) \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_d^n} \Delta \gamma^{p^\top}(s; v) \right] \\
& \geq 0, \quad \forall v(\cdot) \in \mathcal{U}, \quad a.e.a.s.;
\end{aligned} \tag{4.11}$$

4) The following transversality condition holds:

$$\langle \mu, z - EG(y_0, y(T)) \rangle \geq 0, \quad \forall z \in Q. \tag{4.12}$$

Proof

Step 1. Applying Ekeland's variational principle. We first consider the case that the set \mathcal{U}_{ad} is bounded in $L^{\infty,8}_{\mathcal{F},p}[[0, T]; \mathbb{R}^m]$; the unbounded case can be reduced to the bounded case. Assume that

$$\mathcal{U}_{ad} \text{ is bounded in } L^{\infty,8}_{\mathcal{F},p}[[0, T]; \mathbb{R}^m] \tag{4.13}$$

An applications of Ekeland's variational principle will lead to the reduction of a general end-constraint problem to a family of free end-constraint problems.

Define the following auxiliary function

$$J(v(\cdot), x_0; \varepsilon, \delta) = \Psi(J(v(\cdot), x_0), EG(x_0, x(T)); \varepsilon, \delta) \tag{4.14}$$

with $\Psi(\cdot, \cdot; \varepsilon, \delta)$ being defined as in (4.6). Then consider the metric space $(\mathbb{R}^m \times \mathcal{U}_{ad}, d)$ with the distance d defined by

$$d((x_1, v_1(\cdot)), (x_2, v_2(\cdot))) = \sqrt{|x_1 - x_2|^2 + \hat{d}^2(v_1(\cdot), v_2(\cdot))}. \tag{4.15}$$

Tang and Li [34] verify that $\Psi(\cdot, \cdot; \varepsilon, \delta)$ is complete and $J(v(\cdot), x_0; \varepsilon, \delta)$ is continuous and bounded. Also, we have for any given $\varepsilon > 0$,

$$\begin{aligned}
& \Phi(J(v(\cdot), x_0), EG(x_0, x(T)); \varepsilon) > 0, \quad \forall (x_0, v(\cdot)) \in \mathbb{R}^m \times \mathcal{U}_{ad}; \\
& \Phi(J(v(\cdot), y_0), EG(y_0, y(T)); \varepsilon) = \varepsilon; \\
& J(v(\cdot), x_0; \varepsilon, \delta) > 0, \quad \forall (x_0, v(\cdot)) \in \mathbb{R}^m \times \mathcal{U}_{ad}, \\
& \text{for sufficiently small } \delta > 0; \\
& J(u(\cdot), y_0; \varepsilon, \delta) \leq \varepsilon + 2\delta + \inf_{(x_0, v(\cdot)) \in \mathbb{R}^m \times \mathcal{U}_{ad}} J(v(\cdot), x_0; \varepsilon, \delta)
\end{aligned} \tag{4.16}$$

Therefore we can apply Ekeland's variational principle (cf. [10]) and conclude that there exist $u^{\varepsilon\delta} \in \mathcal{U}_{ad}$ and $y_0^{\varepsilon\delta} \in \mathbb{R}^m$ such that

$$\begin{aligned}
1) \quad & J(u^{\varepsilon\delta}(\cdot), y_0^{\varepsilon\delta}; \varepsilon, \delta) \leq \varepsilon + 2\delta; \\
2) \quad & d((y_0^{\varepsilon\delta}, u^{\varepsilon\delta}(\cdot)), (y_0, u(\cdot))) \leq \sqrt{\varepsilon + 2\delta} \\
3) \quad & \bar{J}(v(\cdot), x_0; \varepsilon, \delta) \stackrel{\text{def}}{=} J(v(\cdot), x_0; \varepsilon, \delta) + \sqrt{\varepsilon + 2\delta} d((x_0, v(\cdot)), (y_0^{\varepsilon\delta}, u^{\varepsilon\delta}(\cdot))) \\
& \geq J(u^{\varepsilon\delta}(\cdot), y_0^{\varepsilon\delta}), \quad \forall (x_0, v(\cdot)) \in \mathbb{R}^m \times \mathcal{U}_{ad}.
\end{aligned} \tag{4.17}$$

Set

$$\begin{aligned}
\lambda^{\varepsilon\delta} & \stackrel{\text{def}}{=} \Psi_s(J(u^{\varepsilon\delta}(\cdot), y_0^{\varepsilon\delta}), EG(y_0^{\varepsilon\delta}, y^{\varepsilon\delta}(T)); \varepsilon, \delta), \\
\mu^{\varepsilon\delta} & \stackrel{\text{def}}{=} \Psi_z(J(u^{\varepsilon\delta}(\cdot), y_0^{\varepsilon\delta}), EG(y_0^{\varepsilon\delta}, y^{\varepsilon\delta}(T)); \varepsilon, \delta).
\end{aligned} \tag{4.18}$$

and

$$\begin{aligned}
\lambda^\varepsilon & \stackrel{\text{def}}{=} \lambda^{\varepsilon\delta(\varepsilon)}, \quad \mu^\varepsilon \stackrel{\text{def}}{=} \mu^{\varepsilon\delta(\varepsilon)}, \\
y_0^\varepsilon & \stackrel{\text{def}}{=} y_0^{\varepsilon\delta(\varepsilon)}, \quad u^\varepsilon(\cdot) \stackrel{\text{def}}{=} u^{\varepsilon\delta(\varepsilon)}(\cdot).
\end{aligned}$$

Tang and Li [34] showed that for each sufficiently small $\varepsilon > 0$, we can choose $\delta(\varepsilon) > 0$ such that $\lambda^\varepsilon \geq 0$ and $\mu^\varepsilon \in \mathbb{R}^k$ satisfy the following:

$$\begin{aligned} \lim_{\delta \rightarrow 0^+} (|\lambda^\varepsilon|^2 + |\mu^\varepsilon|^2) &= 1, \\ \langle \mu^\varepsilon, z - EG(y_0^\varepsilon, y^\varepsilon(T)) \rangle &\leq \delta(\varepsilon) \leq \varepsilon. \end{aligned} \quad (4.19)$$

Step 2. Computing the first-order component of the cost variation. In this and the next steps, we look for the necessary conditions for the minimization of $\bar{J}(v(\cdot), x_0; \varepsilon, \delta)$ at $(y_0^\varepsilon, u^\varepsilon(\cdot))$.

For given $(\eta, v(\cdot)) \in \mathbb{R}^m \times \mathcal{U}_{ad}$, set

$$\begin{aligned} u^{\varepsilon\rho}(t) &= u^\varepsilon(t)\chi_{[0,1]\setminus I_\rho}(t) + v(t)\chi_{I_\rho}(t), \\ y_0^{\varepsilon\rho} &= y_0^\varepsilon + |I_\rho|\eta, \\ y^{\varepsilon\rho}(\cdot) &= y(\cdot; u^{\varepsilon\rho}(\cdot), y_0^{\varepsilon\rho}). \end{aligned} \quad (4.20)$$

We introduce, as in (3.4), the following simplified notations:

$$\begin{aligned} \Delta m^\varepsilon(s; v) &\stackrel{\text{def}}{=} m(y^\varepsilon(s), v) - m(y^\varepsilon(s), u^\varepsilon(s)), \\ m^\varepsilon(s) &\stackrel{\text{def}}{=} m(y^\varepsilon(s), u^\varepsilon(s)), \end{aligned} \quad (4.21)$$

with m standing for g, γ, ℓ and all their (up to second-) derivatives in x .

Let $y^{\varepsilon\rho}(\cdot)$ be the solution of (3.1) corresponding to $(y_0^{\varepsilon\rho}, u^{\varepsilon\rho}(\cdot))$. We define, as in (3.9) and (3.10), the half- and first-order processes $y_1^\varepsilon(\cdot), y_2^\varepsilon(\cdot)$, respectively, by

$$\begin{aligned} y_1^\varepsilon(t) &= \int_0^t g_x(y^\varepsilon(s), u^\varepsilon(s)) y_1^\varepsilon(s) ds \\ &\quad + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_d^n} \int_0^t [\gamma_x^p(y^\varepsilon(s), u^\varepsilon(s)) y_1^\varepsilon(s) + \Delta \gamma^{\varepsilon,p}(s; u^{\varepsilon\rho}(s))] dH^p(s) \end{aligned} \quad (4.22)$$

and

$$\begin{aligned} &y_2^\varepsilon(t) \\ &= \int_0^t \left[g_x(y^\varepsilon(s), u^\varepsilon(s)) y_2^\varepsilon(s) + \Delta g^\varepsilon(s; u^{\varepsilon\rho}(s)) + \frac{1}{2} g_{xx}(y^\varepsilon(s), u^\varepsilon(s)) y_1^\varepsilon(s) y_1^\varepsilon(s) \right] ds \\ &\quad + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_d^n} \int_0^t \left[\gamma_x^p(y^\varepsilon(s), u^\varepsilon(s)) y_2^\varepsilon(s) + \frac{1}{2} \gamma_{xx}^p(y^\varepsilon(s), u^\varepsilon(s)) y_1^\varepsilon(s) y_1^\varepsilon(s) \right] dH^p(s) \\ &\quad + \int_0^t \Delta g_x^\varepsilon(s; u^{\varepsilon\rho}(s)) y_1^\varepsilon(s) ds + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_d^n} \int_0^t \Delta \gamma_x^{\varepsilon,p}(s, u^{\varepsilon\rho}(s)) y_1^\varepsilon(x) dH^p(s) + |I_\rho|\eta \end{aligned} \quad (4.23)$$

From Lemma 3.1, we can have

$$\begin{aligned} \sup_{0 \leq t \leq T} E|y_1^\varepsilon(t)|^8 &= O(|I_\rho|^4), \\ \sup_{0 \leq t \leq T} E|y_2^\varepsilon(t)|^8 &= O(|I_\rho|^4), \\ \sup_{0 \leq t \leq T} E|y^{\varepsilon\rho}(t) - y^\varepsilon(t) - y_1^\varepsilon(t) - y_2^\varepsilon(t)|^2 &= o(|I_\rho|^4), \\ &\text{as } |I_\rho| \rightarrow 0. \end{aligned} \quad (4.24)$$

In this step, we are to calculate the first-order component of the cost variation.

From 3) in (4.17) of Step 1, we have

$$\begin{aligned}
& -|I_\rho| \sqrt{\varepsilon + 2\delta} \sqrt{1 + |\eta|^2} \\
& \leq J(u^{\varepsilon\rho}(\cdot), y^{\varepsilon\rho}(0); \varepsilon) - J(u^\varepsilon(\cdot), y_0^\varepsilon; \varepsilon) \\
& \leq \lambda^\varepsilon [J(u^{\varepsilon\rho}(\cdot), y_0^\varepsilon + |I_\rho|\eta) - J(u^\varepsilon(\cdot), y_0^\varepsilon)] \\
& \quad + \sum_{j=1}^m \mu^{\varepsilon j} [EG^j(y_0^\varepsilon + |I_\rho|\eta, y^{\varepsilon\rho}(T)) - EG^j(y_0^\varepsilon, y^\varepsilon(T))] \\
& \quad + O(|J(u^{\varepsilon\rho}(\cdot), y_0^\varepsilon + |I_\rho|\eta) - J(u^\varepsilon(\cdot), y_0^\varepsilon)|^2) \\
& \quad + \sum_{j=1}^m O(|EG^j(y_0^\varepsilon + |I_\rho|\eta, y^{\varepsilon\rho}(T)) - EG^j(y_0^\varepsilon, y^\varepsilon(T))|^2)
\end{aligned} \tag{4.25}$$

Using (4.24), we have

$$\begin{aligned}
& J(u^{\varepsilon\rho}(\cdot), y_0^\varepsilon + |I_\rho|\eta) - J(u^\varepsilon(\cdot), y_0^\varepsilon) \\
= & |I_\rho| < E h_y(y_0^\varepsilon, y^\varepsilon(T)), \eta > + E < h_x(y_0^\varepsilon, y^\varepsilon(T)), y_1^\varepsilon(T) + y_2^\varepsilon(T) > \\
& + \frac{1}{2} E y_1^{\varepsilon\top}(T) h_{xx}(y_0^\varepsilon, y^\varepsilon(T)) y_1^\varepsilon(T) \\
& + E \int_0^T \ell_x(y^\varepsilon(s), u^\varepsilon(s)) [y_1^\varepsilon(s) + y_2^\varepsilon(s)] ds + \frac{1}{2} E \int_0^T y_1^{\varepsilon\top}(s) \ell_{xx}(y^\varepsilon(s), u^\varepsilon(s)) y_1^\varepsilon(s) ds \\
& + E \int_0^T \Delta \ell^\varepsilon(s, u^{\varepsilon\rho}(s)) ds + o(|I_\rho|)
\end{aligned} \tag{4.26}$$

and similarly

$$\begin{aligned}
& EG^j(y_0^\varepsilon + |I_\rho|\eta, y^{\varepsilon\rho}(T)) - EG^j(y_0^\varepsilon, y^\varepsilon(T)) \\
= & |I_\rho| < EG_y^j(y_0^\varepsilon, y^\varepsilon(T)), \eta > + E < G_x^j(y_0^\varepsilon, y^\varepsilon(T)), y_1^\varepsilon(T) + y_2^\varepsilon(T) > \\
& + \frac{1}{2} E y_1^{\varepsilon\top}(T) G_{xx}^j(y_0^\varepsilon, y^\varepsilon(T)) y_1^\varepsilon(T) + o(|I_\rho|)
\end{aligned} \tag{4.27}$$

From Lemma 2.2, we see that

$$\begin{aligned}
-dp^\varepsilon(t) &= \left[g_x^\top(y^\varepsilon(t), u^\varepsilon(t)) p^\varepsilon(t) \right. \\
& \quad \left. + \sum_{d=1}^\infty \sum_{p \in \mathbb{N}_d^n} \gamma_x^p(y^\varepsilon(t), u^\varepsilon(t))^\top J^{p,\varepsilon}(t) + \lambda^\varepsilon \ell_x(y^\varepsilon(t), u^\varepsilon(t)) \right] dt \\
& \quad - \sum_{d=1}^\infty \sum_{p \in \mathbb{N}_d^n} J^{p,\varepsilon}(t) dH^p(t) \\
p^\varepsilon(T) &= \lambda^\varepsilon h_x(y_0^\varepsilon, y^\varepsilon(T)) + \sum_{j=1}^k \mu^{\varepsilon j} G_x^j(y_0^\varepsilon, y^\varepsilon(T)).
\end{aligned} \tag{4.28}$$

and

$$\begin{aligned}
-dP^\varepsilon(t) &= \left[g_x^\top(y^\varepsilon(t), u^\varepsilon(t)) P^\varepsilon(t) + P^\varepsilon(t) g_x(y^\varepsilon(t), u^\varepsilon(t)) \right. \\
& \quad \left. + \sum_{d=1}^\infty \sum_{p \in \mathbb{N}_d^n} \gamma_x^p(y^\varepsilon(t), u^\varepsilon(t))^\top P^\varepsilon(t) \gamma_x^p(y^\varepsilon(t), u^\varepsilon(t)) + \sum_{d=1}^\infty \sum_{p \in \mathbb{N}_d^n} \gamma_x^p(y^\varepsilon(t), u^\varepsilon(t))^\top R^{p,\varepsilon}(t) \right. \\
& \quad \left. + \sum_{d=1}^\infty \sum_{p \in \mathbb{N}_d^n} R^{p,\varepsilon}(t) \gamma_x^p(y^\varepsilon(t), u^\varepsilon(t)) + H_{xx}(y^\varepsilon(t), u^\varepsilon(t), \lambda^\varepsilon, p^\varepsilon(t), J^\varepsilon(t)) \right] dt \\
& \quad - \sum_{d=1}^\infty \sum_{p \in \mathbb{N}_d^n} R^{p,\varepsilon}(t) dH^p(t) \\
P^\varepsilon(T) &= \lambda^\varepsilon h_{xx}(y_0^\varepsilon, y^\varepsilon(T)) + \sum_{j=1}^k \mu^{\varepsilon j} G_{xx}^j(y_0^\varepsilon, y^\varepsilon(T)).
\end{aligned} \tag{4.29}$$

have unique solutions $(p^\varepsilon(\cdot), \{J^{p,\varepsilon}(\cdot)\}_{p \in \mathbb{N}^n})$ and $(P^\varepsilon(\cdot), \{R^{p,\varepsilon}(\cdot)\}_{p \in \mathbb{N}^n})$ respectively, with $p^\varepsilon(\cdot)$ and $P^\varepsilon(\cdot)$ being cadlag processes.

Using Itô's formula, we have from (4.22), (4.28) and (4.29), that

$$\begin{aligned}
& E < \lambda^\varepsilon h_x(y_0^\varepsilon, y^\varepsilon(T)) + \sum_{j=1}^k \mu^{\varepsilon j} G_x^j(y_0^\varepsilon, y^\varepsilon(T)) + \int_0^T \lambda^\varepsilon \ell_x(y^\varepsilon(s), u^\varepsilon(s)), y_1^\varepsilon(T) + y_2^\varepsilon(T) > \\
& = E < p^\varepsilon(T), y_1^\varepsilon(T) + y_2^\varepsilon(T) > \\
& = < p^\varepsilon(0), \eta > |I_\rho| + E \int_0^T (p(s), \Delta g^\varepsilon(s, u^{\varepsilon\rho}(s))) ds \\
& \quad + \sum_{d=1}^\infty \sum_{p \in \mathbb{N}_d^n} E \int_0^T (J^p(s), \Delta \gamma^{\varepsilon,p}(s, u^{\varepsilon\rho}(s))) ds \\
& \quad + \frac{1}{2} E \int_0^T (p(s), g_{xx}(y^\varepsilon(s), u^\varepsilon(s)) y_1^\varepsilon(s) y_1^\varepsilon(s)) ds \\
& \quad + \frac{1}{2} \sum_{d=1}^\infty \sum_{p \in \mathbb{N}_d^n} E \int_0^T (J^p(s), \gamma^p(y^\varepsilon(s), u^\varepsilon(s)) y_1^\varepsilon(s) y_1^\varepsilon(s)) ds \\
& \quad + \sum_{d=1}^\infty \sum_{p \in \mathbb{N}_d^n} E \int_0^T (J^p(s), \Delta \gamma_x^{\varepsilon,p}(s, u^{\varepsilon\rho}(s)) y_1^\varepsilon(s)) ds
\end{aligned} \tag{4.30}$$

Applying Ito's formula to the matrix-valued processes

$$Y(s) = y_1(s) y_1^\top(s) = \begin{pmatrix} y_1^1 y_1^1 & \cdots & y_1^1 y_1^m \\ \vdots & \vdots & \vdots \\ y_1^1 y_1^m & \cdots & y_1^m y_1^m \end{pmatrix}$$

we have

$$\begin{aligned}
dY(t) &= \left[Y(t) g_x^\top(t) + g_x(t) Y(t) + \sum_{d=1}^\infty \sum_{p \in \mathbb{N}_d^n} \gamma_x^p(t) Y(t) \gamma_x^p(t)^\top + \Phi^\varepsilon(t) \right] dt \\
&\quad + \sum_{d=1}^\infty \sum_{p \in \mathbb{N}_d^n} \left[Y(t) \gamma_x^p(t)^\top + \gamma_x^p(t) Y(t) + \Omega^{p,\varepsilon}(t) \right] dH^p(t)
\end{aligned} \tag{4.31}$$

where

$$\begin{aligned}
\Phi^\varepsilon(t) &= \sum_{d=1}^\infty \sum_{p \in \mathbb{N}_d^n} \gamma_x^p(t) y_1(t) \Delta \gamma^p(t, u^\varepsilon(t))^\top + \sum_{d=1}^\infty \sum_{p \in \mathbb{N}_d^n} \Delta \gamma^p(t, u^\varepsilon(t)) y_1(t)^\top \gamma_x^p(t)^\top \\
&\quad + \sum_{d=1}^\infty \sum_{p \in \mathbb{N}_d^n} \Delta \gamma^p(t, u^\varepsilon(t)) \sum_{d=1}^\infty \sum_{p \in \mathbb{N}_d^n} \Delta \gamma^p(t, u^\varepsilon(t))^\top \\
\Omega^{p,\varepsilon}(t) &= y_1(t) \Delta \gamma^p(t, u^\varepsilon(t))^\top + \Delta \gamma^p(t, u^\varepsilon(t)) y_1(t)^\top \\
&\quad + \sum_{d=1}^\infty \sum_{p \in \mathbb{N}_d^n} \Delta \gamma^p(t, u^\varepsilon(t)) \sum_{d=1}^\infty \sum_{p \in \mathbb{N}_d^n} \Delta \gamma^p(t, u^\varepsilon(t))^\top
\end{aligned}$$

and

$$\begin{aligned}
& \lambda^\varepsilon \mathbb{E} y_1^{\varepsilon, \top}(T) h_{xx}(y_0^\varepsilon, y^\varepsilon(T)) y_1^\varepsilon(T) + \sum_{j=1}^k \mu^{\varepsilon, j} \mathbb{E} y_1^{\varepsilon, \top}(T) G_{xx}^j(y_0^\varepsilon, y^\varepsilon(T)) y_1^\varepsilon(T) \\
&= \text{tr} E[P^\varepsilon(T) y_1^\varepsilon(T) y_1^{\varepsilon, \top}(T)] \\
&= -E \int_0^T y_1^{\varepsilon, \top}(s) H_{xx}(y^\varepsilon(s), u^\varepsilon(s), \lambda^\varepsilon, p^\varepsilon(s), J^\varepsilon(s)) y_1^\varepsilon(s) ds \\
&\quad + E \int_0^T \text{tr} P^\varepsilon(s) \left[\sum_{d=1}^\infty \sum_{p \in \mathbb{N}_d^n} \Delta \gamma^{\varepsilon, p}(s; u^{\varepsilon p}(s)) \sum_{d=1}^\infty \sum_{p \in \mathbb{N}_d^n} \Delta \gamma^{\varepsilon, p, \top}(s; u^{\varepsilon p}(s)) \right] ds \\
&\quad + E \int_0^T \text{tr} \left[\sum_{d=1}^\infty \sum_{p \in \mathbb{N}_d^n} R^{p, \varepsilon}(s) \right]^\top \left[\sum_{d=1}^\infty \sum_{p \in \mathbb{N}_d^n} \Delta \gamma^{\varepsilon, p}(s; u^{\varepsilon p}(s)) \sum_{d=1}^\infty \sum_{p \in \mathbb{N}_d^n} \Delta \gamma^{\varepsilon, p, \top}(s; u^{\varepsilon p}(s)) \right] ds \\
&\quad + 2E \int_0^T \text{tr} P^\varepsilon(s) \left[\sum_{d=1}^\infty \sum_{p \in \mathbb{N}_d^n} \gamma_x^{\varepsilon, p}(s) y_1^\varepsilon(s) \Delta \gamma^{\varepsilon, p, \top}(s; u^{\varepsilon p}(s)) \right] ds
\end{aligned} \tag{4.32}$$

Noting the estimates (4.24), we conclude from (4.25)-(4.27) and (4.30)-(4.32) that

$$\begin{aligned}
E &< \lambda^\varepsilon h_y(y_0^\varepsilon, y^\varepsilon(T)) + \sum_{j=1}^k \mu^{\varepsilon, j} G_y^j(y_0^\varepsilon, y^\varepsilon(T) + p^\varepsilon(0), \eta) > |I_\rho| \\
&\quad + \int_0^T \ell^\varepsilon(s, u^{\varepsilon p}) ds + o(|I_\rho|) \geq -|I_\rho| \sqrt{\varepsilon + 2\delta(\varepsilon)} \sqrt{1 + |\eta|^2}
\end{aligned} \tag{4.33}$$

where $\ell^\varepsilon(\cdot; v)$ is defined by

$$\begin{aligned}
\ell^\varepsilon(s; v) &=: E(H(y(s), u^\varepsilon(s), \lambda, p(s), J(s)) - H(y(s), u(s), \lambda, p(s), J(s))) \\
&\quad + \frac{1}{2} E \text{tr} P^\varepsilon(s) \left[\sum_{d=1}^\infty \sum_{p \in \mathbb{N}_d^n} \Delta \gamma^{\varepsilon, p}(s; u^{\varepsilon p}(s)) \sum_{d=1}^\infty \sum_{p \in \mathbb{N}_d^n} \Delta \gamma^{\varepsilon, p, \top}(s; u^{\varepsilon p}(s)) \right] \\
&\quad + \frac{1}{2} E \text{tr} \left[\sum_{d=1}^\infty \sum_{p \in \mathbb{N}_d^n} R^{p, \varepsilon}(s) \right]^\top \left[\sum_{d=1}^\infty \sum_{p \in \mathbb{N}_d^n} \Delta \gamma^{\varepsilon, p}(s; u^{\varepsilon p}(s)) \sum_{d=1}^\infty \sum_{p \in \mathbb{N}_d^n} \Delta \gamma^{\varepsilon, p, \top}(s; u^{\varepsilon p}(s)) \right]
\end{aligned} \tag{4.34}$$

Step 3. Differentiability. For given $v(\cdot) \in \mathcal{U}_{ad}$, applying Lemma 3.2 to the real valued Lebesgue integrable function, we know that there exists $I_\rho \subset [0, T]$ such that

$$\begin{aligned}
|I_\rho| &= \rho, \\
\int_{I_\rho} \ell^\varepsilon(s, v(s)) ds &= \rho \int_0^T \ell^\varepsilon(s; v(s)) ds + o(\rho), \quad \text{as } \rho \rightarrow 0
\end{aligned} \tag{4.35}$$

Next choose the above I_ρ in (4.20), and we have

$$\int_{I_\rho} \ell^\varepsilon(s, v(s)) ds = \rho \int_0^T \ell^\varepsilon(s; u^{\varepsilon p}(s)) ds \tag{4.36}$$

From (4.33)-(4.36), we conclude for given $v(\cdot) \in \mathcal{U}_{ad}$ that

$$\begin{aligned}
E &< \lambda^\varepsilon h_y(y_0^\varepsilon, y^\varepsilon(T)) + \sum_{j=1}^k \mu^{\varepsilon, j} G_y^j(y_0^\varepsilon, y^\varepsilon(T) + p^\varepsilon(0), \eta) > \rho + \rho \int_0^T \ell^\varepsilon(s, v(s)) ds \\
&\geq -\rho \sqrt{\varepsilon + 2\delta(\varepsilon)} \sqrt{1 + |\eta|^2} + o(\rho), \quad \text{as } \rho \rightarrow 0.
\end{aligned} \tag{4.37}$$

Hence

$$\begin{aligned}
E &< \lambda^\varepsilon h_y(y_0^\varepsilon, y^\varepsilon(T)) + \sum_{j=1}^k \mu^{\varepsilon, j} G_y^j(y_0^\varepsilon, y^\varepsilon(T) + p^\varepsilon(0), \eta) > + \int_0^T \ell^\varepsilon(s, v(s)) ds \\
&\geq -\sqrt{\varepsilon + 2\delta(\varepsilon)} \sqrt{1 + |\eta|^2}, \quad \forall \eta \in \mathbb{R}^m \quad \forall v(\cdot) \in \mathcal{U}_{ad}.
\end{aligned} \tag{4.38}$$

This implies that

$$\begin{aligned}
\lambda^\varepsilon E h_y(y_0^\varepsilon, y^\varepsilon(T)) + \sum_{j=1}^k \mu^{\varepsilon, j} E G_y^j(y_0^\varepsilon, y^\varepsilon(T) + p^\varepsilon(0)) &\leq C \sqrt{3\varepsilon}, \\
\int_0^T \ell^\varepsilon(s, v(s)) ds &\geq -\sqrt{\varepsilon + 2\delta(\varepsilon)}, \quad \forall v(\cdot) \in \mathcal{U}_{ad}.
\end{aligned} \tag{4.39}$$

Step 4. Passing to the limit. Without loss of generality, we assume that $\lambda^\varepsilon \rightarrow \lambda, \mu^\varepsilon \rightarrow \mu$, as $\varepsilon \rightarrow 0+$. Let $\varepsilon \rightarrow 0+$. Equation (4.19)₂ gives the following:

$$\begin{aligned} & E \int_0^T (H(y(s), u^\varepsilon(s), \lambda, p(s), J(s)) - H(y(s), u(s), \lambda, p(s), J(s))) ds \\ & + \frac{1}{2} E \int_0^T \text{tr} P^\varepsilon(s) \left[\sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_d^n} \Delta \gamma^{\varepsilon, p}(s; u^{\varepsilon p}(s)) \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_d^n} \Delta \gamma^{\varepsilon, p, \top}(s; u^{\varepsilon p}(s)) \right] ds \\ & + \frac{1}{2} E \int_0^T \text{tr} \left[\sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_d^n} R^{\varepsilon, p}(s) \right]^\top \left[\sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_d^n} \Delta \gamma^{\varepsilon, p}(s; u^{\varepsilon p}(s)) \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_d^n} \Delta \gamma^{\varepsilon, p, \top}(s; u^{\varepsilon p}(s)) \right] ds \\ & \geq 0, \quad \forall v(\cdot) \in U_{ad}; \end{aligned} \quad (4.40)$$

this implies (4.11). Furthermore, (4.11) is obtained from (4.19)₁, (4.9)₂ is obtained from (4.39)₁, and the rest of Theorem 4.1 is checked from (4.28) and (4.29).

Step 5. The unbounded case of \mathcal{U}_{ad} in $L^{\infty, 8}_{\mathcal{F}, p}[[0, T]; \mathbb{R}^m]$. The proof procedure is the same as the step 5 in Tang and Li [34].

The proof of Theorem 4.1 is complete. \square

5. Conclusions

In this paper, necessary maximum principle for optimal control of stochastic system driven by multidimensional Teugel's martingales is proved, where the multidimensional Teugel's martingales are constructed by orthogonalizing the multidimensional Lévy processes. The control variable is allowed to enter the coefficients of the Teugel's martingales, and the control domain is nonconcave. The technique for proving the maximum principle and the obtained result are almost similar to Peng [26] and Tang and Li [34].

References

- [1] J.S.Bras, R.J.Elliott, M.Kohlmann, The partially observed stochastic minimum principle. *SIAM J. Control Optim.* **27** (1989) 1279–1292.
- [2] K.Bahlali, M.Eddahbi, E.Essaky, BSDE associated with Lévy processes and application to PDIE. *Journal of Applied Mathematics and Stochastic Analysis*. **16**(1) (2003) 1–17.
- [3] A.Bensoussan, *Lectures on Stochastic Control*, in Nonlinear Filtering and Stochastic Control, Lecture Notes in Mathematics 972, Proceedings, Cortona, Springer-Verlag, Berlin, New York, 1981.
- [4] J.S.E.Bertoin, *Lévy processes*. Cambridge University Press, Cambridge, 1996.
- [5] J.M.Bismut, An introductory approach to duality in optimal stochastic control. *SIAM Rev.* **20** (1978) 62–78.
- [6] R.K.Boel, Optimal control of jump processes. *Electronics Research Lab. Memo M448*, University of California, Berkeley, CA, July 1974.
- [7] R.K.Boel, P.Varaiya, Optimal control of jump processes. *SIAM J. Control Optim.* **15** (1977) 92–119.
- [8] A.Cadenillas, A stochastic maximum principle for systems with jumps, with applications to finance. *Systems and Control Letters*. **47** (2002) 433–444.
- [9] M.Davis, R.Elliott, Optimal control of jump processes. *Z. Wahrscheinlichkeitstheorie verw. Gebiete*. **40** (1977) 183–202.
- [10] I.Ekeland, Nonconvex minimization problems, *Bull. Amer. Math. Soc.(NS)*. **1** (1979) 443–474.
- [11] C.F.Dunkl, Y. Xu, *Orthogonal Polynomials of Several Variables*. Encyclopedia of Mathematics and its Applications 81, Cambridge University Press, Cambridge, 2001.
- [12] W.H.Fleming, Optimal continuous-parameter stochastic control. *SIAM Rev.* **11** (1969) 470–509.
- [13] N.C.Framstad, B.Øksendal, A.Sulem, Sufficient stochastic maximum principle for the optimal control of jump diffusions and applications to finance. *Journal of Optimization Theory and Applications*. **1** (2004) 77–98.
- [14] U.G.Haussmann, The maximum principle for optimal control of diffusions with partial information. *SIAM J. Control Optim.* **25** (1987) 341–361.
- [15] Y.Hu, Maximum principle of optimal control for Markov processes. *Acta Mathematica Sinica*. **33** (1990) 43–56.
- [16] Y.Hu, S.Peng, Maximum principle for semilinear stochastic evolution control systems. *Stochastics Stochastics Rep.* **33** (1990) 159–180.
- [17] H.J.Kushner, Necessary conditions for continuous parameter stochastic optimization problems. *SIAM J. Control*. **10** (1972) 550–565.
- [18] X.Li, Y.Yao, Maximum principle of distributed parameter systems with time lags, *Distributed Parameter Systems*, Lecture Notes in Control and Information Sciences, **75**, Springer-Verlag, New York, (1985) 410–427.
- [19] J.Z.Lin, Chaotic and predictable representations for multidimensional Lévy processes. <http://arxiv.org>, arXiv:1111.0124, 2011.
- [20] J.Z.Lin, Backward stochastic differential equations and Feynman-Kac formula for multidimensional Lévy processes, with applications in Finance, <http://arxiv.org>, arXiv:1201.6614, 2012.
- [21] Q.X.Meng, M.N.Tang, Necessary and sufficient conditions for optimal control of stochastic systems associated with Lévy processes. *Science in China Series F: Information Sciences*. **52**(2009) 1982–1992.

- [22] D.Nualart,W.Schoutens, Chaotic and predictable representations for Lévy processes. *Stochastic Processes and their Applications*. **90** (2000) 109–122.
- [23] D.Nualart,W.Schoutens, BSDE's and Feynman-Kac Formula for lévy processes with applications in finance. *Bernoulli*. **7** (2001) 761–776.
- [24] K.Mitsui,Y.Tabata, A stochastic linear-quadratic problem with Lévy processes and its application to finance. *Stochastic Processes and their Applications*. **118** (2008) 120–152.
- [25] E.Pardoux,S.Peng, Adapted Solution of a Backward stochastic differential equation. *Systems and Control Letters*. **14** (1990) 55–61.
- [26] S.Peng, A general stochastic maximum principle for optimal control problems. *SIAM J. Control and Optimization*. **28(4)** (1990) 966–979.
- [27] Y.Ren, Reflected backward stochastic differential equations driven by a Lévy process. *ANZIAM J.* **50** (2009) 486–500.
- [28] R.Rishel, A minimum principle for controlled jump processes. *Lecture Notes in Economics and Mathematical Systems*. **107** (1975) 493-508. Springer-Verlag, Berlin, Heidelberg, New York.
- [29] K.Sato, *Lévy processes and infinitely divisible distributions* . Cambridge University Studies in Advanced Mathematics, Vol. 68.Cambridge University Press,Cambridge, 1999.
- [30] J.Shi,Z.Wu, Maximum principle for forward-backward stochastic control system with random jumps and applications to finance. *J Syst Sci Complex*. **2010** (2010) 219–231.
- [31] R.Situ, A maximum principle for optimal controls of stochastic systems with random jumps. *Proc. National Conference on Control Theory and Its Applications*. Qingdao, Shandong, PRC, 1991.
- [32] H.Tang,Z.Wu, Stochastic differential equations and stochastic linear quadratic optimal control problem with Lévy processes. *J Syst Sci Complex*. **22** (2009) 122–136.
- [33] M.N.Tang,Q.Zhang, Optimal variational principle for backward stochastic control systems associated with Lévy processes. *Science China–Mathematics*. **55(4)** (2012) 745–761.
- [34] S.J.Tang,X.J.Li, Necessary conditions for optimal control of stochastic systems with random jumps. *SIAM J. Control and Optimization*. **32(5)** (1994) 1447–1475.
- [35] J.Yong,X.Zhou, *Stochastic Controls:Hamiltonian Systems and HJB Equations* . Springer, New York, 1999.